

Iwasawa λ_5 and μ_5 -invariants of a totally real cubic field with discriminant 1396

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Abstract

In this paper, we will treat a totally real non-cyclic cubic field k with discriminant $1396 = 2^2 \cdot 349$, which is unique up to isomorphism. Then the prime 5 splits completely in k . First we will introduce our previous results on Iwasawa invariants. And, by using these results, we will show that the Iwasawa λ_5 and μ_5 -invariants of k vanish.

Key words : Iwasawa invariants (岩澤不変量), totally real cubic fields (総実 3 次代数体), \mathbb{Z}_p -extensions (\mathbb{Z}_p -拡大)

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1. Introduction

For a number field k and a prime number p , let k_∞ be the cyclotomic \mathbb{Z}_p -extension of k with n -th layer k_n . Let A_n be the p -Sylow subgroup of the ideal class group of k_n . Then there exist integers λ, μ and ν , depending only on k and p , such that $\#A_n = p^{\lambda n + \mu p^n + \nu}$ for sufficiently large n (cf. [Iw59], and also an excellent text book [Wa82]), where $\#G$ denotes the order of a finite group G . The integers $\lambda = \lambda_p(k)$, $\mu = \mu_p(k)$ and $\nu = \nu_p(k)$ are called the (cyclotomic) Iwasawa invariants of k for p . It is conjectured that both $\lambda_p(k)$ and $\mu_p(k)$ always vanish for any totally real number field k and any prime number p (cf. [Gr76], and also [Iw73]). This is called

as Greenberg's conjecture. It is known by a theorem of Iwasawa [Iw56] that if p does not split in k and the class number of k is not divided by p , then Iwasawa $\lambda_p(k)$, $\mu_p(k)$ and $\nu_p(k)$ -invariants vanish. In particular, Greenberg's conjecture is valid for $k = \mathbb{Q}$, the field of rational numbers. Further, for any prime number p , it is shown by Ferrero and Washington [FW79] that the Iwasawa $\mu_p(k)$ -invariant always vanishes if k is an abelian number field, but it is not known yet for the Iwasawa $\lambda_p(k)$ -invariants of totally real number fields k , even if k has a low degree except when $k = \mathbb{Q}$.

Until now, several authors investigated Greenberg's conjecture in the case where k is a real abelian number field (cf. Greenberg [Gr76], Fukuda and Komatsu

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[FK86], Fukuda and the author [FT95], Ichimura and Sumida [IS96, IS97], Kraft and Schoof [KS97], Kurihara [Ku99], and the author [Ta96]). For instance, when $p = 3$, it is shown in [IS96] and [IS97] that the λ_3 -invariants of real quadratic fields $\mathbb{Q}(\sqrt{m})$ vanish for all positive integers $m < 10,000$. Also, Ono [On99] and Byeon [By01, By03] proved that, for any prime number $p \geq 5$, there are infinitely many real quadratic fields k with $\lambda_p(k) = \mu_p(k) = \nu_p(k) = 0$ by estimating the number of such k .

Concerning cubic fields, we gave some affirmative computational data for totally real cubic fields (including cyclic cubic fields) and $p = 3$ (cf. [Ta99a]), for cyclic cubic fields and $p = 5, 7$ (cf. [Ta99b]), in the case where a given prime p splits completely. In this paper, we will treat a totally real non-cyclic cubic field k with discriminant $1396 = 2^2 \cdot 349$, which is unique up to isomorphism. Then the prime 5 splits completely in k . First, we will recall our previous results (cf. [Gr76], [Ta99a]). After that, we will calculate the order of some subgroups of the intermediate fields of the cyclotomic \mathbb{Z}_5 -extension of k , and finally show by using the previous results that Iwasawa invariants λ_5 and μ_5 of k vanish.

2. Previous results

In this section, we will recall our previous results which we use in the next section. Let Γ be the Galois group of k_∞ over k , and let A_n^Γ be the subgroup of A_n consisting of ideal classes which are invariant under the action of Γ , namely, A_n^Γ is the Γ -invariant part of A_n . Let ν_p be the p -adic valuation normalized by $\nu_p(p) = 1$. In the case where p splits completely in k , the following theorem, which is proved in [Ta99a], holds.

Theorem 2.1 Let k be a totally real number field and p an odd prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p . Then, for every sufficiently large n ,

$$\#A_n^\Gamma = \#A_0 p^{\nu_p(R_p(k)) - [k:\mathbb{Q}] + 1},$$

where $R_p(k)$ is the p -adic regulator of k and $[k:\mathbb{Q}]$ the degree of k over \mathbb{Q} .

Let D_n is the subgroup of A_n consisting of ideal classes represented by products of prime ideals of k_n lying above p . It is clear that $D_n \subset A_n^\Gamma$. By using Theorem 2.1, we obtain the following alternative formulation of a theorem of Greenberg [Gr76, Theorem 2] on the vanishing of the Iwasawa invariants.

Theorem 2.2 Let k be a totally real number field and p an odd prime number. Assume that p splits completely in k and also that Leopoldt's conjecture is valid for k and p . Then the following conditions are equivalent:

- (1) $\lambda_p(k) = \mu_p(k) = 0$,
- (2) $D_n = \#A_0 p^{\nu_p(R_p(k)) - [k:\mathbb{Q}] + 1}$ for some $n \geq 0$.

In particular, if $\nu_p(R_p) = [k:\mathbb{Q}] - 1$ and if $A_0 = D_0$, then $\lambda_p(k) = \mu_p(k) = 0$.

3. Example

In this section, we will study a totally real cubic field k defined by $f(x) = x^3 - x^2 - 7x + 5$, which is a non-Galois extension over \mathbb{Q} (i.e., the Galois group of its Galois closure is the symmetric group of degree 3). This k is unique up to isomorphism, and also the prime 5 splits completely in k . Our purpose is to show that $\lambda_5(k) = \mu_5(k) = 0$ by applying Theorems 2.1 and 2.2.

Our computation has been carried out by means of excellent number theoretic calculator packages "KASH 3" [KASH3] and "GP/PARI Ver.2.7.0" [PARI2]. Also, we use the polynomials generating totally real cubic fields in a table made by M. Olivier, which is available at the site of "GP/PARI". Note that most of the previous effective methods to verify Greenberg's conjecture have been developed in the case where p is an odd prime number and k is a real abelian number field such that $[k:\mathbb{Q}]$ divides $p - 1$ (cf. [Gr76], [FK86], [FT95], [IS96], [IS97], [KS97]).

Now, we will give computational data of the total real cubic field k in which $p = 5$ splits completely, and show that $\lambda_5(k) = \mu_5(k) = 0$. Note that this k is the only one example such that k is a non-Galois cubic extension with $p = 5$ splitting completely and with discriminant less than 2000.

Example 3.1 Let k be a totally real cubic field defined by $f(x) = x^3 - x^2 - 7x + 5$ which is unique up to isomorphism. Then the discriminant of k is $1396 = 2^2 \cdot 349$ and $p = 5$ splits completely in k . Let θ be a root of $f(x) = 0$ and θ' one of its conjugates. By using KASH 3, we see that a system of fundamental units of k is

$$\{4 - 7\theta + 2\theta^2, 8 - \theta^2\}$$

and the class number of k is 1. Put $\varepsilon_1 = 4 - 7\theta + 2\theta^2$ and $\varepsilon_2 = 8 - \theta^2$. Further, put $\varepsilon'_1 = 4 - 7\theta' + 2\theta'^2$ and $\varepsilon'_2 = 8 - \theta'^2$, which are conjugates of ε_1 and ε_2 respectively. Since we may take the following values as θ and θ' (other pairs are possible and we obtain the same conclusion on the order of A_n^Γ and D_n for any other pairs):

$$\begin{aligned}\theta &\equiv 177579 \pmod{5^{10}}, \\ \theta' &\equiv 734132 \pmod{5^{10}},\end{aligned}$$

we obtain

$$\begin{aligned}\varepsilon_1 &\equiv 953183 \pmod{5^{10}}, \\ \varepsilon_2 &\equiv 8667517 \pmod{5^{10}}, \\ \varepsilon'_1 &\equiv 3822928 \pmod{5^{10}}, \\ \varepsilon'_2 &\equiv 5284709 \pmod{5^{10}}.\end{aligned}$$

Taking the 5-adic logarithms of these, we get

$$\begin{aligned}\log_5 \varepsilon_1 &\equiv 8024605 \pmod{5^{10}}, \\ \log_5 \varepsilon_2 &\equiv 2861705 \pmod{5^{10}}, \\ \log_5 \varepsilon'_1 &\equiv 5566195 \pmod{5^{10}}, \\ \log_5 \varepsilon'_2 &\equiv 4923115 \pmod{5^{10}}.\end{aligned}$$

Hence it follows that

$$R_5(k) \equiv 4 \cdot 5^2 \pmod{5^3}.$$

Thus, we have $v_5(R_5(k)) = 2$. In particular, Leopoldt's conjecture is valid in this case. Now, by Theorem 2.1, we obtain

$$\#A_n^\Gamma = \#A_0 \cdot 5^{v_5(R_5(k)) - [k:\mathbb{Q}] + 1} = 1$$

for all integers $n \geq 0$, which implies that $\#D_n = 1$ for all integers $n \geq 0$. Hence it follows from Theorem 2.2 that $\lambda_5(k) = \mu_5(k) = 0$.

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